The Beckenbach Inequality and Its Inverse

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In this paper, a general Beckenbach inequality and its inverse are given. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let (X, Σ, μ) be a finite measure space and $L_p = L_p(X, \Sigma, \mu)$ be the space of all pth power nonnegative integrable functions over (X, Σ, μ). If $p > 1$, $1/p + 1/q = 1$, and $f \in L_p$, $g \in L_q$, then $fg \in L_1$ and the Hölder inequality

$$\|fg\|_1 \leqslant \|f\|_p \, \|g\|_q \tag{1}$$

holds, where $\|f\|_p = (\int_X f^p \, d\mu)^{1/p}$ etc. Equality in (1) holds if and only if $\alpha f^p = \beta g^q$ a.e. for some nonzero constants α and β.

As is well known, there are several generalizations of the Hölder inequality (see, e.g., [2, 4]). One of them is now known as the Beckenbach inequality [1]. In [3, 6–9], various proofs for the Beckenbach inequality are given, and the continuous version and some variants of the inequality are obtained. We follow [9], which says:

Let positive real α, β, A, B and a positive continuous function k defined on the positive part of the real line be given and let p and q satisfy $p^{-1} + q^{-1} = 1$. Then for $p > 1$, the inequality

$$G(f) \geqslant G(h), \tag{2}$$

where

$$G(f) = \left(\alpha A + \beta \int_0^T f^p \, dt\right)^{1/p} \Big/ \left(\alpha B + \beta \int_0^T fk \, dt\right)$$

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and

$$h(t) = [Ak(t)/B]^{q/p}, \qquad 0 \leqslant t \leqslant T,$$

holds for any continuous function f on $0 \leqslant t < T < \infty$. The sign of inequality in (2) is reversed for $0 < p < 1$.

Motivated by [9], we introduce a general form of the Beckenbach inequality and give it a simple proof in Section 2, and then give its inverse inequality in Section 3. Of course, the inverse inequality includes as special cases many inverse Hölder inequalities.

2. Beckenbach Inequality

THEOREM 1. *Suppose (X, Σ, μ), L_p, p, and q as above. Then for any $f \in L_p$, $g \in L_q$, and positive numbers a, b, c, the inequality*

$$\frac{(a + c\int_X f^p \, d\mu)^{1/p}}{b + c\int_X fg \, d\mu} \geqslant \frac{(a + c\int_X h^p \, d\mu)^{1/p}}{b + c\int_X hg \, d\mu} \tag{3}$$

holds, where $h = (ag/b)^{q/p}$. The sign of equality in (3) holds if and only if $f = h$ a.e. The sign of inequality in (3) is reversed if $0 < p < 1$.

Proof. Obviously, $h \in L_p$. Noting that $1 + q/p = q$, the right-hand side of (3) becomes

$$\frac{(a + c\int_X (ag/b)^q \, d\mu)^{1/p}}{b + c\int_X (ag/b)^{q/p} g \, d\mu} = \frac{(a/b)^{q/p} \, (a(b/a)^q + c\int_X g^q \, d\mu)^{1/p}}{(a/b)^{q/p} \, (b(b/a)^{q/p} + c\int_X g^q \, d\mu)}$$
$$= \left(a^{-q/p} b^q + c\int_X g^q \, d\mu\right)^{-1/q}.$$

By the Hölder inequality (1) and its discrete version, we have

$$b + c\int_X fg \, d\mu \leqslant b + c\left(\int_X f^p \, d\mu\right)^{1/p} \left(\int_X g^q \, d\mu\right)^{1/q}$$
$$= a^{1/p}(ba^{-1/p}) + \left(c\int_X f^p \, d\mu\right)^{1/p} \left(c\int_X g^q \, d\mu\right)^{1/q}$$
$$\leqslant \left(a + c\int_X f^p \, d\mu\right)^{1/p} \left(a^{-q/p} b^q + c\int_X g^q \, d\mu\right)^{1/q}$$

which is just (3). The equality holds if and only if $\alpha f^p = \beta g^q$ a.e. for some nonzero α and β and $a/(a^{-q/p} b^q) = \int_X f^p \, d\mu / \int_X g^q \, d\mu$. Hence the equality holds if and only if $\alpha f^p = \beta g^q$ a.e. and $(a/b)^q = \beta/\alpha$ which is equivalent to $f^p = h^p$ a.e.; that is, $f = h$ a.e. This completes the proof of Theorem 1.

In particular, let $a = b$ and $c = 1$; (3) becomes

$$a + \int_X fg \, d\mu \leqslant \left(a + \int_X f^p \, d\mu\right)^{1/p} \left(a + \int_X g^q \, d\mu\right)^{1/q}$$

for any positive number a and we have (1) if let $a \to 0$.

Remark 1. The condition $1/p + 1/q = 1$ $(p > 1)$ in Theorem 1 can be replaced with $1/p + 1/q = 1/r$ $(p, q > 0)$ which is equivalent to $(p/r)^{-1} + (q/r)^{-1} = 1$. In fact, substituting p/r, q/r, a^r, b^r, f^r, g^r, and h^r for p, q, a, b, f, g, and h, we have

$$\frac{(a^r + c\int_X f^p \, d\mu)^{1/p}}{(b^r + c\int_X f^r g^r \, d\mu)^{1/r}} \geqslant \frac{(a^r + c\int_X h^p \, d\mu)^{1/p}}{(b^r + c\int_X h^r g^r \, d\mu)^{1/r}}, \tag{4}$$

whenever $f \in L_p$, $g \in L_q$, $1/p + 1/q = 1/r$, and a, b, c, p, $q > 0$. Equality holds if and only if $f = h$ a.e. Similar to the method in [5] we know that (4) holds whenever $1/p + 1/q = 1/r$ and p, $-q$, $-r > 0$, and the sign of inequality in (4) is reversed if $1/p + 1/q = 1/r$ and either p, $-q$, $r > 0$ or $-p$, $-q$, $-r > 0$.

Remark 2. According to Theorem 1, we noted that the statements "the sign of equality (in (2)) holds iff f is a constant function" and (2) holds "for any continuous function f" are not true in Theorem 2 and 4 of [9]. For example, if $\alpha = \beta = A = B = T = 1 = -f$ and $p = k = 2$ then (2) does not hold. We also noted that either A should be replaced by A^r or $h(t) = (a^{1/r}k/B)^{q/p}$ in Theorem 4 of [9]. There is similar negligence in Theorem 3 of the same paper.

If $X = \{m+1, m+2, ..., n\}$ and μ is chosen to be the counting measure on X, then $L_p = l_p$ and $f \in L_p$ is a finite sequence $x = (x_{m+1}, x_{m+2}, ..., x_n)$, where $x_{m+1}, x_{m+2}, ..., x_n$ are nonnegative. In this case we obtain a discrete analogue of Theorem 1 as follows:

THEOREM 2. *Suppose that a, b, $c > 0$, x_i, $y_i \geqslant 0$, $z_i = (ay_i/b)^{q/p}$ $(i = m+1, m+2, ..., n)$. Let p and q satisfy $p^{-1} + q^{-1} = 1$ and $p > 1$. Then*

$$\frac{(a + c\sum_{i=m+1}^n x_i^p)^{1/p}}{b + c\sum_{i=m+1}^n x_i y_i} \geqslant \frac{(a + c\sum_{i=m+1}^n z_i^p)^{1/p}}{b + c\sum_{i=m+1}^n y_i z_i}. \tag{5}$$

Equality holds if and only if $x_i = z_i$ $(i = m+1, m+2, ..., n)$.

Of course, (5) can also proved directly from Höler's inequality as Theorem 1:

$$b+c\sum_{i=m+1}^{n} x_i y_i = a^{1/p}(ba^{-1/p}) + \sum_{i=m+1}^{n} (c^{1/p}x_i)(c^{1/q}y_i)$$
$$\leqslant \left(a+c\sum_{i=m+1}^{n} x_i^p\right)^{1/p} \left(a^{-q/p}b^q + c\sum_{i=m+1}^{n} y_i^q\right)^{1/q}$$
$$= \left(a+c\sum_{i=m+1}^{n} x_i^p\right)^{1/p} \frac{(a+c\sum_{i=m+1}^{n} z_i^p)^{1/p}}{b+c\sum_{i=m+1}^{n} y_i z_i}.$$

Remark 3. To prove Theorems 1 and 2, we can also use the more fundamental inequality

$$x^{1/p}y^{1/q} \leqslant \frac{x}{p}+\frac{y}{q} \qquad \left(x, y \geqslant 0, \frac{1}{p}+\frac{1}{q}=1, p>1\right). \tag{6}$$

For example, put $x = a/A$ and $y = b^q a^{-q/p}/B$ and then put $x = cx_i^p/A$ and $y = cy_i^q/B$ ($i = m+1, m+2, ..., n$), where

$$A = a + c\sum_{i=m+1}^{n} x_i^p, \qquad B = a^{-q/p}b^q + c\sum_{i=m+1}^{n} y_i^q.$$

Apply (6) to obtain some inequalities and then add these inequalities to obtain (5).

3. Inverse Beckenbach Inequality

To establish an inverse Beckenbach inequality, we need the following results.

THEOREM 3 [10]. *Let* (X, Σ, μ), L_p, p, q, f, *and* g *be the same as Theorem* 1. *If* f *and* g *satisfy respectively* $0 < m_1 \leqslant f(x) \leqslant M_1$ *and* $0 < m_2 \leqslant g(x) \leqslant M_2$ *for almost all* $x \in X$, *then for any two positive numbers* α *and* β *we have*

$$\|f\|_p \, \|g\|_q \leqslant c_{p,q} \, \|fg\|_1,$$

where

$$c_{p,q} = (\alpha p)^{-1/p} \, (\beta q)^{-1/q} \max\left\{\frac{\alpha M_1^p + \beta m_2^q}{M_1 m_2}, \frac{\alpha m_1^p + \beta M_2^q}{m_1 M_2}\right\}. \tag{7}$$

Equality holds if and only if $\mu(E \cup F) = \mu(X)$ *and*

$$\mu(E) = \frac{(\alpha p M_1^p - \beta q m_2^q)\,\mu(X)}{\alpha p(M_1^p - m_1^p) + \beta q(M_2^q - m_2^q)}, \tag{8}$$

where

$$E = \{x \in X: f(x) = m_1,\ g(x) = M_2\}, \quad F = \{x \in X: f(x) = M_1,\ g(x) = m_2\}. \tag{9}$$

Moreover, $c_{p,q}$*, as a function of* α *and* β*, attains its minimum whenever*

$$\alpha = M_1 m_2 M_2^q - m_1 m_2^q M_2, \qquad \beta = m_1 M_1^p M_2 - m_1^p M_1 m_2.$$

THEOREM 4 [10]. *Suppose that* $0 < m_1 \leqslant x_k \leqslant M_1$, $0 < m_2 \leqslant y_k \leqslant M_2$ $(k = 1, 2, ..., n)$, $p > 1$ *and* $1/p + 1/q = 1$. *Then for all* $\alpha, \beta > 0$ *we have*

$$\left(\sum_{k=1}^n x_k^p\right)^{1/p} \left(\sum_{k=1}^n y_k^q\right)^{1/q} \leqslant c_{p,q} \sum_{k=1}^n x_k y_k, \tag{10}$$

where $c_{p,q}$ *is the same as* (7). *The sign of equality holds if and only if*

$$K = \frac{(\alpha p M_1^p - \beta q m_2^q) n}{\alpha p(M_1^p - m_1^p) + \beta q(M_2^q - m_2^q)} \tag{11}$$

is an integer, and K's $x_k = m_1$, $y_k = M_2$, *others* $x_k = M_1$, $y_k = m_2$.

Now we state a discrete version and a continuous version of an inverse Beckenbach inequality.

THEOREM 5. *Suppose that* $p > 1$, $1/p + 1/q = 1$, a, b, c, x_i, $y_i > 0$ *and* $z_i = (ay_i/b)^{q/p}$ $(i = m+1, m+2, ..., n)$. *If there are positive numbers* m_1, M_1, m_2, *and* M_2, *such that*

$$m_1 \leqslant a^{1/p} \leqslant M_1, \qquad m_1 \leqslant c^{1/p} x_i \leqslant M_1,$$
$$m_2 \leqslant a^{-1/p} b \leqslant M_2, \qquad m_2 \leqslant c^{1/q} y_i \leqslant M_2$$

$(i = m+1, m+2, ..., n)$, *then for all* $\alpha, \beta > 0$ *we have*

$$\frac{(a + c\sum_{i=m+1}^n x_i^p)^{1/p}}{b + c\sum_{i=m+1}^n x_i y_i} \leqslant c_{p,q} \frac{(a + c\sum_{i=m+1}^n z_i^p)^{1/p}}{b + c\sum_{i=m+1}^n y_i z_i}, \tag{12}$$

where $c_{p,q}$ *is the same as* (7). *Equality holds if and only if*

$$K = \frac{(\alpha p M_1^p - \beta q m_2^q)(n - m + 1)}{\alpha p(M_1^p - m_1^p) + \beta q(M_2^q - m_2^q)}$$

is an integer, and K's $a_k = m_1$, $b_k = M_2$, *others* $a_k = M_1$, $b_k = m_2$, $k = m, m+1, ..., n$, *where* $a_m = a^{1/p}$, $b_m = a^{-1/p}b$, $a_i = c^{1/p}x_i$, $b_i = c^{1/q}y_i$ $(i = m+1, m+2, ..., n)$.

THEOREM 6. *Let* $a, b, c > 0$, $p > 1$, $1/p + 1/q = 1$, $f \in L_p$, $g \in L_q$, *and* $h = (ag/b)^{q/p}$. *Let* m_1, M_1, m_2, *and* M_2 *be positive numbers such that* $m_1 \leqslant a^{1/p} \leqslant M_1$, $m_2 \leqslant a^{-1/p}b \leqslant M_2$ *and*

$$m_1 \leqslant c^{1/p}f(x) \leqslant M_1, \qquad m_2 \leqslant c^{1/q}g(x) \leqslant M_2$$

for almost all $x \in X$. *Then for all* $\alpha, \beta > 0$ *we have*

$$\frac{(a + c\int_X f^p \, d\mu)^{1/p}}{b + c\int_X fg \, d\mu} \leqslant c_{p,q} \frac{(a + c\int_X h^p \, d\mu)^{1/p}}{b + c\int_X hg \, d\mu}, \tag{13}$$

where $c_{p,q}$ *is the same as* (7). *The conditions to hold equality are the same as in Theorem* 3 *except that either* $a^{1/p} = m_1$, $a^{-1/p}b = M_2$, *or* $a^{1/p} = M_1$, $a^{-1/p}b = m_2$.

Proof. Inequality (13) can be proved from (12) by the usual methods which are stated in [2, Section 6.4]. Hence we only prove (12) as follows: Similar to the proof of Theorem 1, we have

$$\frac{(a + c\sum_{i=m+1}^n z_i^p)^{1/p}}{b + c\sum_{i=m+1}^n y_i z_i} = \frac{1}{(a^{-q/p}b^q + c\sum_{i=m+1}^n y_i^q)^{1/q}}.$$

By (10), we obtain

$$\begin{aligned}\left(a + c \sum_{i=m+1}^n x_i^p\right)^{1/p} &\left(a^{-q/p}b^q + c \sum_{i=m+1}^n y_i^q\right)^{1/q} \\ &\leqslant c_{p,q}\left(a^{1/p}(a^{-1/p}b) + \sum_{i=m+1}^n (c^{1/p}x_i)(c^{1/q}y_i)\right) \\ &= c_{p,q}\left(b + c \sum_{i=m+1}^n x_i y_i\right).\end{aligned}$$

This is just (12). The conditions to hold equality of (12) can follow from Theorem 4.

Remark 4. Similarly to Remark 1, if $1/p + 1/q = 1/r$ $(p, q > 0)$ then

$$\frac{(a^r + c\int_X f^p \, d\mu)^{1/p}}{(b^r + c\int_X f^r g^r \, d\mu)^{1/r}} \leqslant c_{p,q}^r \frac{(a^r + c\int_X h^p \, d\mu)^{1/p}}{(b^r + c\int_X h^r g^r \, d\mu)^{1/r}},$$

where a, b, c, f, and g are the same as Theorem 6, except that $m_1 \leqslant a^{r/p} \leqslant M_1$, $m_2 \leqslant a^{-r/p} b \leqslant M_2$,

$$c^r_{p,q} = (\alpha p)^{-1/p} (\beta q)^{-1/q} \times r^{1/r} \max\left\{\frac{(\alpha M_1^p + \beta m_2^q)^{1/r}}{M_1 m_2}, \frac{(\alpha m_1^p + \beta M_2^q)^{1/r}}{m_1 M_2}\right\}. \tag{14}$$

In particular, if $a = b$, $c = 1$, and $a \to 0$, then the inverse Hölder inequality (see, e.g., [10])

$$\|f\|_p \|g\|_q \leqslant c^r_{p,q} \|fg\|_r \tag{15}$$

holds, where $c^r_{p,q}$ attains its minimum whenever

$$\alpha = \alpha_0 = M_1^r m_2^r M_2^q - m_1^r m_2^q M_2^r, \qquad \beta = \beta_0 = m_1^r M_1^p M_2^r - m_1^p M_1^r m_2^r.$$

Remark 5. In [5], there are two forms for $c^r_{p,q}$ which are given in (14),

$$c^r_{p,q} = \frac{\{r(M_1^{p/2} M_2^{q/2} + m_1^{p/2} m_2^{q/2})\, M(p, q, r)\}^{1/r}}{\{p(m_2 M_2)^{q/2}\}^{1/p} \{q(m_1 M_1)^{p/2}\}^{1/q}}, \tag{16}$$

where $M(p, q, r) = \max\{s^{p/2-r} t^{q/2-r} \mid s = m_1, M_1, t = m_2, M_2\}$, and

$$c^r_{p,q} = \frac{\{r(M_1^p M_2^q - m_1^p m_2^q)\}^{1/r}}{\{p(M_1^r m_2^r M_2^q - m_1^r M_2^r m_2^q)\}^{1/p} \{q(m_1^r M_1^p M_2^r - M_1^r m_1^p m_2^r)\}^{1/q}}. \tag{17}$$

It is easy to check that (17) is just (14) when $\alpha = \alpha_0$ and $\beta = \beta_0$. Thus we will conclude that (17) is better than (16) if we can show that (16) is a special case of (14) for some α and β.

In fact, note that $p/2 - r \geqslant 0$ if and only if $q/2 - r \leqslant 0$, and so either

$$M_1^{p/2-r} M_2^{q/2-r} \leqslant M_1^{p/2-r} m_2^{q/2-r} \quad \text{or} \quad M_1^{p/2-r} M_2^{q/2-r} \leqslant m_1^{p/2-r} M_2^{q/2-r},$$

similarly to $m_1^{p/2-r} m_2^{q/2-r}$. Hence

$$M(p, q, r) = \max\{M_1^{p/2-r} m_2^{q/2-r}, m_1^{p/2-r} M_2^{q/2-r}\}.$$

Now it is not difficult to verify that (16) is a special case of (14) whenever

$$\alpha = \frac{(m_2 M_2)^{q/2}}{m_1^{p/2} m_2^{q/2} + M_1^{p/2} M_2^{q/2}}, \qquad \beta = \frac{(m_1 M_1)^{p/2}}{m_1^{p/2} m_2^{q/2} + M_1^{p/2} M_2^{q/2}}.$$

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